DECOMPOSITION THEOREMS AND KERNEL THEOREMS FOR A CLASS OF FUNCTIONAL SPACES

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Abstract. We prove new theorems about properties of generalized functions defined on Gelfand-Shilov spaces S^β with $0 \le \beta < 1$. For each open cone $U \subset \mathbb{R}^d$ we define a space $S^\beta(U)$ which is related to $S^\beta(\mathbb{R}^d)$ and consists of entire analytic functions rapidly decreasing inside U and having order of growth $\le 1/(1-\beta)$ outside the cone. Such sheaves of spaces arise naturally in nonlocal quantum field theory, and this motivates our investigation. We prove that the spaces $S^\beta(U)$ are complete and nuclear and establish a decomposition theorem which implies that every continuous functional defined on $S^\beta(\mathbb{R}^d)$ has a unique minimal closed carrier cone in $\mathbb{R}^d$. We also prove kernel theorems for spaces over open and closed cones and elucidate the relation between the carrier cones of multilinear forms and those of the generalized functions determined by these forms.

1. Introduction

In this paper we investigate the angular localizability property of generalized functions belonging to the spaces S'^β, $0 \le \beta < 1$. This property was revealed in applications of these classes of generalized functions to nonlocal quantum field theory. The test function spaces S^β and S^β_α were introduced by Gelfand and Shilov [1]. If $\beta < 1$, they consist of entire functions, and continuous linear functionals on these spaces are analytic in the sense that they are representable by Taylor series convergent in the topology (weak as well as strong) of the dual spaces S'^β and S'^β_α.

To develop a nonlocal field theory, one should use certain spaces related to S^β, S^β_α and associated with cones in $\mathbb{R}^d$. The notion of a minimal carrier cone of an analytical functional plays a key role in these applications. The existence of such a quasi-support follows from appropriate decomposition theorems for spaces over cones. For spaces with two indices, the corresponding analysis is performed in [2]. It is based on the fact that these spaces belong to the well-studied class of DFS spaces, that is, spaces dual to FS (Fréchet-Schwartz) spaces. The properties of DFS spaces are reviewed, for example, in the survey [3]. The topological structure of the spaces $S^\beta(U)$, where U is an open cone in $\mathbb{R}^d$, is more complicated than that of $S^\beta_\alpha(U)$. They are not DFS spaces, and even the proof of their completeness is a challenge. If $\beta = 0$, then proving decomposition theorems presents additional difficulties, and one way around them is to use methods of the theory of hyperfunctions. This limiting case is of prime interest because the space S^0 is nothing but the Fourier transform of the Schwartz space $\mathcal{D}$ of all infinitely differentiable compactly supported functions. The existence of smallest carrier cones for elements of the dual space S'^0 can be established in a roundabout way [4] by restricting these functionals to S^0_α. But this result alone is not sufficient for applications, and the properties of these function spaces call for further investigation. In this paper, particular attention is given to the extension of the theory to multilinear forms, including the derivation of kernel theorems for $S^\beta(U)$ and for spaces over closed cones, which are constructed from spaces over open cones by means of the inductive limit.

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We note that spaces of type S^{β} over cones arise naturally when one extends the theory of Fourier-Laplace transformation to analytic functionals, and especially when one generalizes the Paley-Wiener-Schwartz theorem [5, 6]. Vladimirov's version [5] of this theorem shows that there is an isomorphism between the space of tempered distributions supported in a properly convex closed cone K and an algebra of analytic functions defined on a certain tubular domain and growing at most polynomially. If we relax the bound on the growth of functions at infinity and pose the question of finding the class of functionals corresponding to the enlarged algebra, then we inevitably arrive at the spaces $S'^{\beta}(K)$. If the bound is removed altogether, then we arrive at $S'^{0}(K)$.

The paper is organized as follows. The next section contains basic definitions and some preliminary information on the function spaces in question. In Sect. 3 we use Palomodov's criterion [7] to prove that these spaces are complete. In Sects. 4, 5 we prove the main decomposition theorem for spaces over cones. This theorem implies that the correspondence $K \to S^{\beta}(K)$ is a lattice (anti-)homomorphism. For $\beta > 0$, the proof is simpler and uses the non-triviality of the space $S^{\beta}_{1-\beta}$. The proof for $\beta = 0$ is given in Sect. 5 and relies on Hörmander's estimates [8]. Since the weight functions in these estimates are plurisubharmonic, we develop a general technique (Appendix 1) for approximating the indicator functions (that determine the spaces under study) by plurisubharmonic functions. In Sect. 6 we prove that every element of $S'^{\beta}(\mathbb{R}^d)$ has a unique minimal closed carrier cone. In Sect. 7 we establish that the spaces associated with open and closed cones are nuclear and indicate some consequences of this result. The corresponding kernel theorems, which enable one to identify multilinear separately continuous forms on these spaces with linear functionals, are proved in Sect. 8. The method devised for this purpose is also applicable to other spaces of analytic functions. In Sect. 9 we show that the study of analytic functionals generated by multilinear forms leads naturally to the notion of a strong carrier cone. The difference between the notions of a carrier cone and a strong carrier cone is elucidated in Appendix 2. In Sect. 10 we derive a Paley-Wiener-Schwartz-type theorem, which precisely describes the properties of the Laplace transforms of functionals that belong to S'^{β} and are strongly carried by a properly convex cone.

The theorems of the theory of linear topological spaces used below are contained in [9, 10, 11]. We refer to [5] for the basic facts about plurisubharmonic functions.

2. Basic definitions and preliminaries

Let $0 \le \beta < 1$ and let U be an open cone in $\mathbb{R}^d$. We define $S^{\beta,b}(U)$ to be the intersection (projective limit) of the Hilbert spaces $H^{\beta,B}_N(U)$, $B > b$, $N = 0, 1, 2, \dots,$ that consist of entire functions on $\mathbb{C}^d$ and are equipped with the inner products

$$\langle f, g\rangle_{U,B,N} = \int \overline{f(z)}g(z) \prod_{j=1}^{d}(1+|x_j|)^{2N} \exp\{-2d(Bx,U)^{1/(1-\beta)} - 2|By|^{1/(1-\beta)}\}\, d\lambda, \quad (1)$$

where $z = x + iy$, $d(x,U) = \inf_{\xi\in U} |x - \xi|$ is the distance from x to U and $d\lambda = dx\,dy$ is the Lebesgue measure on $\mathbb{C}^n$. Note that $d(Bx,U) = Bd(x,U)$ because U is a cone. It is clear from this definition that $S^{\beta,b}(U)$ is a Fréchet space (that is, a complete metrizable space). We denote the union of $S^{\beta,b}(U)$ over all $b > 0$ by $S^{\beta}(U)$ and endow it with the inductive limit topology. This space is independent of the choice of the norm $|\cdot|$ on $\mathbb{R}^d$, because all these norms are equivalent. In what follows we use either the Euclidean norm or the $l^{1/(1-\beta)}$–norm

$$\left(\sum_{j=1}^{d} |y_j|^{1/(1-\beta)}\right)^{1-\beta}. \quad (2)$$

The latter is convenient when treating miltilinear forms on $S^0(\mathbb{R}^{d_1})\times\cdots\times S^0(\mathbb{R}^{d_n})$ because then the weight function in (1) has a multiplicative property. Namely, if we put

$$\rho_{U,B,N}(z) = -N\sum_j \ln(1+|x_j|) + d(Bx,U)^{1/(1-\beta)} + |By|^{1/(1-\beta)}, \tag{3}$$

then the function determining the space $S^0(U_1\times\cdots\times U_n)$, where $U_k\subset\mathbb{R}^{d_k}$, is given by the product of the functions of the functions determining the $S^0(U_k)$:

$$\exp\{-\rho_{U_1\times\cdots\times U_n,B,N}(z)\} = \prod_{k=1}^n \exp\{-\rho_{U_k,B,N}(z_k)\}. \tag{4}$$

The space $S^{\beta,b}(U)$ may also be represented as the intersection of the Banach spaces $E_N^{\beta,B}(U)$ of entire functions with the norms

$$\|f\|'_{U,B,N} = \sup_{z\in\mathbb{C}^d} |f(z)|\, e^{-\rho_{U,B,N}(z)}. \tag{5}$$

This is precisely the original definition given in [12], but the reformulation in terms of Hilbert spaces is best suited to most of the questions discussed below. The representation

$$S^{\beta,b}(U) = \bigcap_{B>b,N\geq 0} E_N^{\beta,B}(U)$$

makes it clear that $S^\beta(\mathbb{R}^d)$ coincides with the Gelfand-Shilov space S^β and with the Gurevich space W^Ω, where $\Omega(y) = y^{1/(1-\beta)}$. However, these spaces were not given a topology in [1] and the notion of convergence of sequences was used instead. In Sect. 3 we show that this simplified "sequential" approach agrees with the natural topology described above. We also note that $S^{\beta,b}(U)$ can be treated as a countably normed space $Z(M_p)$ specified by $M_p = \exp\{-\rho_{U,b+1/p,p}\}$ if we omit the condition $M_p(z)\geq C(y)$ from the definition [1] of this class of spaces. The equivalence between the system (5) of norms and the system $\|f\|_{U,B,N}$ defined by the inner products (1) can be established using Cauchy's integral formula, which shows that

$$|f(z)| \leq C\|f\|_{L^2(\mathcal{B})},$$

where $\mathcal{B}$ is any bounded neighborhood of z in $\mathbb{C}^d$. Taking $\mathcal{B} = \{\zeta\colon |z-\zeta|<1\}$ and applying the triangle inequality to every term on the right-hand side of (3), we see that $\rho_{U,B',N}(\zeta)\leq \rho_{U,B,N}(z) + C_{B,B',N}$ for $\zeta\in\mathcal{B}$ and any $B'<B$. Therefore,

$$|f(z)|^2 e^{-2\rho_{U,B,N}(z)} \leq C'\int_{\mathcal{B}} |f(\zeta)|^2 e^{-2\rho_{U,B',N}(\zeta)} \mathrm{d}\lambda \leq C'\|f\|^2_{U,B',N}, \tag{6}$$

On the other hand, it is clear that $\|f\|_{U,B,N}\leq C''\|f\|'_{U,B',N+d+1}$.

For each closed cone $K\subset\mathbb{R}^d$ we define the space $S^\beta(K)$ as the inductive limit of the spaces $S^\beta(U)$, where U runs through the open cones that contain K as a compact subcone. (This is written[1] $K\Subset U$.) All these spaces are continuously embedded into the space $S^\beta(\{0\})$ associated with the degenerate closed cone consisting of one point, namely, the origin. Its elements are entire functions of order $1/(1-\beta)$ and finite type or of order less than $1/(1-\beta)$. It should be noted that we suggest cones for geometric visualization, although we are really dealing with a sheaf of spaces over the sphere compactifying $\mathbb{R}^d$. Although the cone $\{0\}$ is closed in $\mathbb{R}^d$, it corresponds to the empty subset of the sphere, which is both closed and open. Therefore the space $S^\beta(\{0\})$ along with its topology can be defined directly by formula (1) with $d(x,0)=|x|$. As we shall see in Sect. 5, this topology coincides with the inductive topology determined by the injections $S^\beta(U)\to S^\beta(\{0\})$, where U ranges over all open cones in $\mathbb{R}^d$.

[1]For arbitrary cones V_1, V_2, the notation $V_1\Subset V_2$ means that $\bar V_1\setminus\{0\}\subset V_2$. Here and in what follows, we use a bar to denote the closure of a set.

Definition 1. Let v be a continuous linear functional on $S^\beta(\mathbb{R}^d)$. We say that v is *carried* by a closed cone $K \subset \mathbb{R}^d$ if this functional admits a continuous extension to $S^\beta(K)$.

When such an extension exists, it is unique by the following theorem.

Theorem 1. *There is a constant λ (depending only on d and β) such that the space $S^{\beta,\lambda b}(\mathbb{R}^d)$ is dense in $S^{\beta,b}(U)$ in the topology of $S^{\beta,\lambda b}(U)$ for every open cone $U \subset \mathbb{R}^d$ and every $b > 0$. As a consequence, $S^\beta(\mathbb{R}^d)$ is sequentially dense in $S^\beta(U)$ and in any space $S^\beta(K)$, where K is a closed cone.*

Proof. When $\beta > 0$, we can use the fact that the space $S^\beta_{1-\beta}$ is nontrivial. According to [1], there is a $\gamma > 0$ such that for any $A > 0$ the space $S^\beta_{1-\beta}(\mathbb{R}^d)$ contains a nontrivial nonnegative function g_0 satisfying the bound

$$|g_0(z)| \le C \exp\left\{ -\left|\frac{x}{A}\right|^{1/(1-\beta)} + \left|\frac{\gamma y}{A}\right|^{1/(1-\beta)} \right\}. \tag{7}$$

Let $f \in S^{\beta,b}(U)$. We normalize g_0 by the condition $\int g_0(x)\,dx = 1$ and set $f_\nu = f\sigma_\nu$, where $\sigma_\nu(z)$ is a sequence of Riemann sums for the integral $\int g_0(z-\xi)\,d\xi$ or, more explicitly,

$$\sigma_\nu(z) = \sum_{k\in\mathbb{Z}^n, |k|<\nu^2} g_0\left(z - \frac{k}{\nu}\right)\nu^{-n}.$$

Clearly, $f_\nu \in S^\beta(\mathbb{R}^d)$ if $A < 1/b$. The sequence σ_ν converges to 1 in $\mathbb{R}^d$ and we have $|\sigma_\nu(z)| \le C' \exp\{|\gamma y/A|^{1/(1-\beta)}\}$ because the integral sums for $\exp\{-|x/A|^{1/(1-\beta)}\}$ are bounded. Therefore, $\sigma_\nu(z) \to 1$ uniformly on compact sets in $\mathbb{C}^d$ by Vitali's theorem. Moreover, the sequence f_ν is bounded in every norm $|\cdot|'_{U,B,N}$, where $B > b + \gamma/A$. Hence $f_\nu \to f$ in the topology of $S^{b+\gamma/A}(U)$ because it is a Montel space.[2] We see that in this case Theorem 1 holds for any $\lambda > 1 + \gamma$.

If $\beta = 0$, then this argument fails because S^0_1 is trivial. However, Theorem 1 can easily be deduced from an analogous theorem established for $S^0_\alpha(U)$ in [2] by an alternative method using Hörmander's L^2-estimates. The space $S^0_\alpha(U)$ with $\alpha > 1$ is the union of the Banach spaces $E^{0,B}_{\alpha,A}(U)$ of entire functions with the norms

$$\|f\|_{\alpha,U,A,B} = \sup_{z\in\mathbb{C}^d} |f(z)| \exp\left\{ \left|\frac{x}{A}\right|^{1/\alpha} - d(Bx, U) - |By| \right\}. \tag{8}$$

Let us show that $S^0_\alpha(U)$ is dense in $S^0(U)$. Let $f \in S^{0,b}(U)$. The sets

$$\{\tilde f \colon \|f - \tilde f\|'_{U,B,N} \le \delta\}, \quad \text{where } \delta > 0,\ B > b,\ N = 0, 1, \dots,$$

form a base of neighborhoods of f in $S^{0,b}(U)$. We take a function $g \in E^{0,1}_{\alpha,1}(\mathbb{R}^d)$ with $g(0) = 1$ and consider the sequence $f_\nu(z) = f(z)g(z/\nu)$. If $\nu > 1/(B-b)$, then $f_\nu \in E^{0,B}_{\alpha,\nu}(U)$. Let $b < B' < B$. Then we have

$$\|f - f_\nu\|'_{U,B,N} \le \|f\|'_{U,B',N+1} \sup_z \left|1 - g\left(\frac{z}{\nu}\right)\right| (1+|x|)^{-1} e^{-\epsilon|y|},$$

where $\epsilon = B - B' > 0$. If $\nu > 2/\epsilon$, then $|1 - g(z/\nu)| \le C\exp\{\epsilon|y|/2\}$. Given $\delta > 0$, we can choose R such that

$$C\|f\|'_{U,B',N+1} \sup_{|z|\ge R} (1+|x|)^{-1} e^{-\epsilon|y|/2} < \delta.$$

Taking ν large enough for the inequality $\|f\|'_{U,B',N+1} \sup_{|z|<R} |1 - g(z/\nu)| < \delta$ to hold, we obtain $\|f - f_\nu\|'_{U,B,N} < \delta$. Now let $\alpha' > \alpha$. Clearly, $f_\nu \in E^{0,B}_{\alpha',A}(U)$ for any $A > 0$.

[2]In [1], countably normed Montel spaces were termed *perfect spaces* and it was proved that every space $Z(M_p)$ is perfect. We also note that $S^b(U)$ is a Montel space because it is nuclear, see below.

Theorem 5 of [2] shows that if $\mu > 2ed$, then the function f_ν can be approximated in the norm $\|\cdot\|_{\alpha',U,1,\mu B}$ by elements of $E^{0,\mu B}_{\alpha',1}(\mathbb{R}^d)$ with any degree of accuracy. This norm is stronger than $\|\cdot\|_{U,\mu B,N}$. Therefore, in this case Theorem 1 holds for any $\lambda > 2ed$, because then there are $\mu > 2ed$ and $B > b$ such that $\mu B < \lambda b$. □

We note that if $v \in S'^{\beta}$ is carried by a cone K, then the restriction of v to S^{β_1} with $\beta_1 < \beta$ is also carried by this cone. However the converse is not true in general. In what follows, the open mapping theorem is used repeatedly. When dealing with the S^β-type spaces associated with cones, we can use Grothendieck's version [9] of this important theorem (or the even more general version given by Raikov in Appendix 1 to the Russian edition of [10]) because all these spaces, being Hausdorff and inductive limits of sequences of Fréchet spaces, belong to the class[3] $\mathcal{LF}$ and are ultrabornological (spaces of type (β) in the terminology of [9]). In [13], we showed that none of the spaces $S^\beta(U)$, $S^\beta(K)$ except $S^\beta(\{0\})$ are DFS spaces because their dual spaces are non-metrizable.

3. The completeness theorem

The completeness of the spaces $S^\beta(U)$ was proved in [14] using another definition given in terms of real variables. For the reader's convenience, we present an alternative proof starting from the norms (5).

Theorem 2. *The inductive spectrum of the Fréchet spaces $S^{\beta,b}(U)$, $b = 1, 2, \dots$, is acyclic.*

Proof. Let $\mathcal{U}_b$ be the neighborhood of the origin in $S^{\beta,b}(U)$ specified by $\|f\|'_{U,b+1/2,0} < 1/2$. Clearly, $\mathcal{U}_{b_0} \subset \mathcal{U}_b$ for any $b > b_0$. According to Theorem 6.1 of [7], it suffices to verify that the topology on $\mathcal{U}_{b_0}$ induced by that of $S^{\beta,b}(U)$, $b > b_0$, is independent of b. Let $f_0 \in \mathcal{U}_{b_0}$ and $B > b$. We denote by $\mathcal{V}_{B,N,\epsilon}$ the intersection of $\mathcal{U}_{b_0}$ and the neighborhood of f_0 in $S^{\beta,b}(U)$ given by $\|f - f_0\|_{U,B,N} < \epsilon$. We shall show that for any B, B_1 satisfying $B > B_1 > B_0 = b_0 + 1/2$, for every $N_1 \geq 0$, and for every $\epsilon_1 > 0$, there are numbers N and ϵ such that

$$\mathcal{V}_{B,N,\epsilon} \subset \mathcal{V}_{B_1,N_1,\epsilon_1}. \tag{9}$$

This means that the topology induced on $\mathcal{U}_{b_0}$ by that of $S^{\beta,b}(U)$ is not weaker than the topology induced by that of $S^{\beta,b_1}(U)$, where $b_1 < b$ (the reverse is obvious). In what follows we set $\beta = 0$ for simplicity and comment on the case $\beta > 0$ at the end of the proof. If $f \in \mathcal{V}_{B,N,\epsilon}$, then the function $f_1 = f - f_0$ satisfies the estimates

$$|f_1(x)| < e^{B_0 d(x,U) + B_0|y|}, \tag{10}$$

$$|f_1(x)| < \epsilon\,(1 + |x|)^{-N} e^{Bd(x,U)+B|y|}. \tag{11}$$

We claim that, for properly chosen N and ϵ, this implies that

$$|f_1(x)| < \epsilon_1(1 + |x|)^{-N_1} e^{B_1 d(x,U)+B_1|y|}. \tag{12}$$

We introduce the notation $\varepsilon(x) = \epsilon(1+|x|)^{-N}$, $\varepsilon_1(x) = \epsilon_1(1+|x|)^{-N_1}$ and define a number $R(x)$ by the equation

$$e^{B_0 R} = \varepsilon_1 e^{B_1 R}. \tag{13}$$

In the region $d(x,U) + |y| \geq R(x)$, the inequality (12) follows from (10). In the complementary region, (12) follows from (11) if

$$\varepsilon\, e^{BR} = \varepsilon_1\, e^{B_1 R} \tag{14}$$

and, a fortiori, if $\varepsilon\, e^{BR} < \varepsilon_1\, e^{B_1 R}$. Equations (13) and (14) give $\varepsilon = \varepsilon_1^A$, where $A = (B - B_0)/(B_1 - B_0)$. Hence the desired inclusion (9) follows if we take $\epsilon \leq \epsilon_1^A$ and $N \geq AN_1$,

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[3] We note that, in contrast to the definition given in [11], Grothendieck's definition of this class does not require that the inductive limit be strict.

This proof extends to $\beta > 0$ by an obvious change of notation, which yields the same conclusion with the modified number $A = (\tilde{B} - \tilde{B}_0)/(\tilde{B}_1 - \tilde{B}_0)$, where $\tilde{B} = B^{1/(\beta-1)}$. $\square$

By [7], the acyclicity ensures that the following assertions hold.

Corollary 1. *The space $S^\beta(U)$ is Hausdorff and complete. A set $\mathcal{B} \subset S^\beta(U)$ is bounded if and only if it is contained in some space $S^{\beta,B}(U)$ and is bounded in each of its norm.*

It is certainly obvious that $S^\beta(U)$ is a Hausdorff space because its topology is stronger than the topology of uniform convergence. We also note that a linear map of $S^\beta(U)$ (as of any bornological space) to a locally convex space is continuous if and only if it is bounded on bounded sets, which is in turn equivalent to the sequential continuity, see [11].

4. Proof of the decomposition theorem for $\beta > 0$

Here and in Sect. 5 we use the Euclidean norm on $\mathbb{R}^d$. We recall that the intersection of a cone V with the unit sphere is called the *projection* of this cone and is denoted by $\mathrm{pr}\, V$.

Theorem 3. *Let U, U_1, U_2 be open cones in $\mathbb{R}^d$ such that $\bar{U}_1 \cap \bar{U}_2 \Subset U$. Then every function $f \in S^\beta(U)$, $\beta \geq 0$, can be decomposed as $f = f_1 + f_2$, where $f_i \in S^\beta(U \cup U_i)$, $i = 1, 2$.*

Proof. If $\beta > 0$, then we can use the same function $g_0 \in S^\beta_{1-\beta}$ as in the proof of Theorem 1. For simplicity, we assume that f satisfies the estimate

$$|f(z)| \leq C_N(1 + |x|)^{-N} \exp\{d(x, U)^{1/(1-\beta)} + |y|^{1/(1-\beta)}\}, \quad N = 0, 1, 2, \ldots. \tag{15}$$

This does not cause any loss of generality because the spaces involved are invariant under the dilation $f(x) \to f(\lambda x)$, $\lambda > 0$. The hypothesis $\bar{U}_1 \cap \bar{U}_2 \Subset U$ implies that the closed cones $V_1 = \bar{U}_1 \setminus U$, $V_2 = \bar{U}_2 \setminus U$ have disjoint projections. Therefore the distances from $\mathrm{pr}\, V_1$ to V_2 and from $\mathrm{pr}\, V_2$ to V_1 are positive. In the Euclidean metric, these distances coincide. Indeed, if the first distance is attained at points $x_1 \in \mathrm{pr}\, V_1$ and $x_2 \in V_2$, then the equation $|x_1 - x_2|^2 = \big| |x_2| x_1 - |x_2|^{-1} x_2 \big|^2$ implies that $d(\mathrm{pr}\, V_1, V_2) \geq d(\mathrm{pr}\, V_2, V_1)$, and the reverse inequality holds by symmetry. We denote this distance by θ.

We now introduce the auxiliary open cone

$$W = \left\{ \xi \in \mathbb{R}^d \colon d(\xi, V_2) < \frac{\theta}{2}|\xi| \right\}$$

and define $g(z)$ by

$$g(z) = \int_W g_0(z - \xi)\, d\xi. \tag{16}$$

We claim that if the constant A in (7) is small enough, then $gf \in S^\beta(U \cup U_1)$, that is

$$|(gf)(z)| \leq C'_N(1 + |x|)^{-N} \exp\{d(Bx, U \cup U_1)^{1/(1-\beta)} + |By|^{1/(1-\beta)}\} \tag{17}$$

for some $B > 0$. Let $W_1 = \{x \in \mathbb{R}^d \colon d(x, V_2) \geq 3\theta|x|/4\}$. Then

$$|x - \xi| \geq \frac{\theta}{4}|x| \quad \text{for all} \quad x \in W_1,\ \xi \in W. \tag{18}$$

Indeed, if this is not the case, then there are points $x \in W_1$ and $\xi \in W$ such that $|x| = 1$, $|x - \xi| < \theta/4$, and $|\xi| \leq 1$. Also, there is a point $x_2 \in V_2$ such that $|\xi - x_2| < \theta/2$. Then $|x - x_2| < 3\theta/4$ by the triangle inequality. This contradicts the definition of W_1. It follows from (7) and (18) that

$$|g(z)| \leq C_{A'} \exp\left\{ -\left|\frac{\theta x}{4A'}\right|^{1/(1-\beta)} + \left|\frac{\gamma y}{A}\right|^{1/(1-\beta)} \right\} \tag{19}$$

for every $A' > A$. Since $d(x, U) \le |x|$, we see that the function gf decreases in the cone W_1 if $A < \theta/4$ and the inequalities (17) hold in this cone with any $B > 1 + \gamma/A$. On the other hand, we have

$$d(x, V_1) \ge \frac{\theta}{4}|x| \quad \text{for} \quad x \notin W_1 \tag{20}$$

by the triangle inequality. Hence $d(x, V_1) \ge \theta d(x, U)/4$ in this region. Since $d(x, U \cup U_1) = \min\{d(x, U), d(x, U_1 \setminus U)\}$ and $U_1 \setminus U \subset V_1$, we see that the inequalities (17) hold everywhere if we add the condition $B \ge 4/\theta$.

Furthermore, the condition $\int g_0(\xi)\mathrm{d}\xi = 1$ implies that

$$(1 - g)(z) = \int_{\complement W} g_0(z - \xi)\mathrm{d}\xi.$$

Taking $W_2 = \{x \in \mathbb{R}^d \colon d(x, V_2) \le \theta|x|/4\}$, we see that $|x - \xi| \ge \theta|x|/4$ for $x \in W_2$ and $\xi \in \complement W$. On the other hand,

$$d(x, V_2) \ge \frac{\theta}{4}|x| \quad \text{for} \quad x \notin W_2. \tag{21}$$

Therefore $(1-g)f \in S^\beta(U \cup U_2)$ provided that $A < \theta/4$ as before. This proves the theorem for $\beta > 0$. □

5. The use of Hörmander's estimates

Proof of Theorem 3 for $\beta = 0$. We first perform a decomposition into smooth functions satisfying the bounds at infinity that are characteristic of the elements of $S^0(U \cup U_i)$, and then we restore analyticity. Let W, W_1, W_2 be the same auxiliary cones as in the previous section. We take an arbitrary nonnegative function $\chi_0 \in C_0^\infty(\mathbb{R}^d)$ whose integral is 1 and whose support lies in the unit ball, and we set

$$\chi(x) = \int\limits_W \chi_0(x - \xi)\,\mathrm{d}\xi.$$

The argument in Sect. 4 shows that

$$|\chi(x)|e^{d(x,U)} \le Ce^{bd(x,U \cup U_1)}, \tag{22}$$

$$|1 - \chi(x)|e^{d(x,U)} \le Ce^{bd(x,U \cup U_2)}, \tag{23}$$

$$\left|\frac{\partial \chi}{\partial x_j}\right| e^{d(x,U)} \le Ce^{bd(x,U \cup U_1 \cup U_2)}, \quad j = 1, \dots, d, \tag{24}$$

where we can take $4/\theta$ for b. The situation is even simpler than before because $\operatorname{supp}\chi$ is contained in the 1-neighborhood of W and $\chi(x) = 0$ at all points of W_1 lying outside a ball of sufficiently large radius R. Inside the ball, inequality (22) holds with $C = e^R$ and any b. Outside W_1, it holds with $C = 1$ and $b \ge 4/\theta$ by (20). Similarly, (23) follows from the equation $1 - \chi(x) = 0$, which holds for all $x \in W_2$ outside a sufficiently large ball. The derivatives of χ are uniformly bounded, and their supports lie in the 1-neighborhood of the boundary of W. At those points of the supports that lie outside a sufficiently large ball, both (20) and (21) hold, and this yields (24).

We set

$$f = f_1 + f_2, \quad f_1(z) = f(z)\chi(x), \quad f_2(z) = f(z)(1 - \chi(x)).$$

It follows from (22) and (23) that

$$\|f_1\|_{U\cup U_1,b,N} \le C\|f\|_{U,1,N}, \quad \|f_2\|_{U\cup U_2,b,N} \le C\|f\|_{U,1,N}, \qquad N = 0, 1, 2, \dots \tag{25}$$

To obtain an analytic decomposition, we write

$$f = f_1' + f_2', \quad f_1' = f_1 - \psi, \quad f_2' = f_2 + \psi$$

and subject ψ to the equations

$$\frac{\partial \psi}{\partial \bar{z}_j} = \eta_j, \tag{26}$$

where

$$\eta_j \overset{\text{def}}{=} f\frac{\partial \chi}{\partial \bar{z}_j} = \frac{1}{2} f \frac{\partial \chi}{\partial x_j}, \qquad j = 1, \dots, d. \tag{27}$$

By the inequality (6) (which holds even for $B' = B$ in the case $\beta = 0$), the functions $\eta_j(z)$ satisfy

$$|\eta_j(z)| \le C_N \|f\|_{U,1,N}\, e^{\rho_{U\cup U_1 \cup U_2 b,N}(z)}. \tag{28}$$

It remains to show that there is a solution of (26) with the required behavior at infinity. This can be done using Hörmander's L^2-estimates. However, the weight functions in these estimates are given by exponents of plurisubharmonic functions while the indicator functions (3) are not of this form. Therefore we need the following lemma, which is proved in Appendix 1.

Lemma 1. *Let V be an open cone in $\mathbb{R}^d$ and $B > 2edb$. For every function $\eta(z)$, $z \in \mathbb{C}^n$, satisfying the inequalities*

$$|\eta(z)| \le C_N\, e^{\rho_{V,b,N}(z)}, \tag{29}$$

there is a plurisubharmonic function $\varrho(z)$ with values in $(-\infty, +\infty)$ such that

$$|\eta(z)| \le e^{\varrho(z)} \le C'_N\, e^{\rho_{V,B,N}(z)}. \tag{30}$$

In our case, $V = U \cup U_1 \cup U_2$ and we apply Lemma 1 to $\eta = \max |\eta_j|$. Put

$$\tilde{\varrho}(z) = 2\varrho(z) + (d+1)\ln(1 + |z|^2).$$

Then the functions η_j belong to $L^2(\mathbb{C}^d, e^{-\tilde{\varrho}}\mathrm{d}\lambda)$. Their definition (27) implies that the compatibility conditions $\partial \eta_j/\partial \bar{z}_k = \partial \eta_k / \partial \bar{z}_j$ are fulfilled. By Theorem 15.1.2 of [8], the system of equations (26) has a solution ψ such that

$$2\int |\psi|^2 e^{-\tilde{\varrho}}(1+|z|^2)^{-2}\mathrm{d}\lambda \le \int \sum_j |\eta_j|^2 e^{-\tilde{\varrho}}\mathrm{d}\lambda. \tag{31}$$

It follows from (30) and (31) that

$$\psi \in L^2\left(\mathbb{C}^d, e^{-2\rho_{V,B,N-d-3}}\mathrm{d}\lambda\right)$$

for any N. Combining this with estimates (25), we see that $f'_1 \in S^0(U \cup U_1)$ and $f'_2 \in S^0(U \cup U_2)$, as required. Theorem 3 is proved. □

Remark 1. The proof of Theorem 3 shows that if $\bar{U}_1 \cap \bar{U}_2 = \{0\}$, then every element $f \in S^\beta(\{0\})$ may be decomposed as $f = f_1 + f_2$, where $f_i \in S^\beta(U_i)$, $i = 1, 2$. This special case is covered by Theorem 3 with a slightly changed wording, where the words "open cones" are replaced by "cones with open projections". Applying the open mapping theorem, we see that the topology defined on $S^\beta(\{0\})$ by the norms $\|\cdot\|_{\{0\},B,N}$, where $d(x, 0) = |x|$, coincides with the inductive topology determined by the pair of injections $S^\beta(U_i) \to S^\beta(\{0\})$, $i = 1, 2$, as well as with the inductive topology determined by the injections $S^\beta(U) \to S^\beta(\{0\})$, where U ranges over all open cones in $\mathbb{R}^d$.

6. The existence of smallest carrier cones

Theorem 4. *For every continuous linear functional on $S^\beta(\mathbb{R}^d)$, $\beta \geq 0$, there is a unique minimal closed carrier cone $K \subset \mathbb{R}^d$.*

Proof. By Theorem 3, we have

$$S^\beta(K_1 \cap K_2) = S^\beta(K_1) + S^\beta(K_2), \tag{32}$$

for every pair of closed cones in $\mathbb{R}^d$. Indeed, if $K_1 \cap K_2 = \{0\}$, then Remark 1 applies because there are open cones U_i such that $K_i \Subset U_i$ and $\bar{U}_1 \cap \bar{U}_2 = \{0\}$. If $K_1 \cap K_2 \neq \{0\}$ and $f \in S^\beta(U)$ with $U \Supset K_1 \cap K_2$, then there are U_i satisfying $\bar{U}_1 \cap \bar{U}_2 \Subset U$.

Now we show that (32) implies the following dual relation:

$$S'^\beta(K_1 \cap K_2) = S'^\beta(K_1) \cap S'^\beta(K_2), \tag{33}$$

where all spaces are regarded as subspaces of $S'^\beta(\mathbb{R}^d)$. The nontrivial part of relation (33) states that if a functional $v \in S'^\beta(\mathbb{R}^d)$ is carried by K_1 and by K_2, then $K_1 \cap K_2$ is also a carrier cone of v. Let v_i be continuous extensions of v to $S^\beta(K_i)$ and let $f \in S^\beta(K_1 \cap K_2)$. Using the decomposition $f = f_1 + f_2$, where $f_i \in S^\beta(K_i)$, we define an extension of v to $S^\beta(K_1 \cap K_2)$ by $\hat{v}(f) = v_1(f_1) + v_2(f_2)$. This extension is well defined. Indeed, if $f = f_1' + f_2'$ is another decomposition, then

$$f_1 - f_1' = f_2' - f_2 \in S^\beta(K_1) \cap S^\beta(K_2) = S^\beta(K_1 \cup K_2)$$

and hence $v_1(f_1 - f_1') = v_2(f_2' - f_2)$ because $S^\beta(\mathbb{R}^d)$ is dense in $S^\beta(K_1 \cup K_2)$ by Theorem 1. The functional $\hat{v}$ is obviously continuous in the inductive topology $\mathcal{T}$ determined by the injections $S^\beta(K_i) \to S^\beta(K_1 \cap K_2)$, $i = 1, 2$, and this topology coincides with the original topology τ of $S^\beta(K_1 \cap K_2)$ by the open mapping theorem [9]. Indeed, τ is not stronger than $\mathcal{T}$ and $(S^\beta(K), \mathcal{T})$ belongs to the class $\mathcal{LF}$ because both spaces $S^\beta(K_i)$ are in this class and $\mathcal{T}$ coincides with the quotient topology of the outer sum $S^\beta(K_1) \oplus S^\beta(K_2)$ modulo a closed subspace (see [10], Ch. V, Proposition 28).

The relation (33) yields an analogous relation for the intersection of any finite family of closed cones. Then the existence of a smallest carrier cone for every $v \in S'^\beta(\mathbb{R}^d)$ can be established by standard compactness arguments. Indeed, let K be the intersection of all carrier cones of v and let U be an open cone such that $U \Supset K$. The projections of the cones complementary to the carriers cover the compact set $\mathrm{pr}\,\complement U$, and we can choose a finite subcovering $\mathrm{pr}\,\complement K_j$ from this open (in the topology of the unite sphere) covering. Then $\cap_j K_j \Subset U$. Therefore, the functional v is continuous in the topology of $S^\beta(U)$, and K is a carrier cone of v. This proves the theorem. $\square$

Combining (32) with the obvious formula

$$S^\beta(K_1 \cup K_2) = S^\beta(K_1) \cap S^\beta(K_2),$$

we see that the map $K \to S^\beta(K)$ is a lattice (anti-)homomorphism from the lattice of closed cones in $\mathbb{R}^d$ to the lattice of linear subspaces of $S^\beta(\{0\})$. This is equivalent to the exactness of the sequence

$$0 \longrightarrow S^\beta(K_1 \cup K_2) \xrightarrow{i} S^\beta(K_1) \oplus S^\beta(K_2) \xrightarrow{s} S^\beta(K_1 \cap K_2) \longrightarrow 0, \tag{34}$$

where s takes each pair of functions $f_{1,2} \in S^\beta(K_{1,2})$ to the difference of their restrictions to $K_1 \cap K_2$. As shown above, the sequence (34) is even topologically exact at the term $S^\beta(K_1 \cap K_2)$. But we cannot assert this for the term $S^\beta(K_1 \cup K_2)$. In other words, we cannot claim that the original topology of this space coincides with the projective topology determined by the canonical embeddings into $S^\beta(K_i)$, $i = 1, 2$. This differs essentially from the case of the DFS-spaces $S^\beta_\alpha(K)$ considered in [2], where the topological exactness of an analogous sequence evidently follows from the open mapping theorem, which applies

because any finite sum of DFS spaces and any closed subspace of a DFS space also belong to this class. However, the sequence (34) is topologically exact in the important event that $K_1 \cap K_2 = \{0\}$, because $S^\beta(\{0\})$ is a DFS space. Then Theorem 5 of [14] shows that every functional $v \in S'^\beta$ with carrier cone $K_1 \cup K_2$, where $K_1 \cap K_2 = \{0\}$, can be decomposed into a sum of functionals $v_i \in S'^\beta(K_i)$, $i = 1, 2$.

7. Nuclearity

Lemma 2. *For any $B < B'$ and $N > N'$, the natural injection $H_N^{\beta,B}(U) \to H_{N'}^{\beta,B'}(U)$ is a Hilbert-Schmidt map.*

Proof. We use the fact that holomorphic functions are pluriharmonic and satisfy the Laplace equation $\Delta f = 0$, where $\Delta = \sum_j (\partial^2/\partial x_j^2 + \partial^2/\partial y_j^2)$. As before, we write $L^2(\mathbb{C}^d, e^{-2\rho}\mathrm{d}\lambda)$ for the Hilbert space of complex-valued functions on $\mathbb{C}^d$ that are square-integrable with the weight $\exp\{-2\rho_{U,B,N}\}$, where $\rho_{U,B,N}$ is defined by (3). In what follows we omit the subscripts U, B, N and write ρ' for the function specified by U, B', N'. The space $H_N^{\beta,B}(U)$ is a close subspace of $L^2(\mathbb{C}^d, e^{-2\rho}d\lambda)$ and hence is separable. We need an auxiliary function belonging to the space $S_\alpha^{1-\alpha}$, where $\alpha > \beta$. If $\beta < 1/2$, then the function $e^{-t^2} \in S_{1/2}^{1/2}$ is suitable. As shown in [1], § IV.8, every space $S_\alpha^{1-\alpha}(\mathbb{R})$ with $1/2 < \alpha < 1$ contains an element of the form $\psi(t^2)$, where $\psi \not\equiv 0$ is an entire function having exponential growth of order $1/(2\alpha)$ in the complex plane and exponential decrease of the same order along the real semi-axis $t > 0$. We assume that $\psi(0) = 1$. Let $p \in \mathbb{R}^d$, $q \in \mathbb{R}^d$ and $\Psi(p, q) = \psi(p^2 + q^2)$. According to [1], § IV.9, we have $\Psi \in S_\alpha^{1-\alpha}(\mathbb{R}^{2d})$ and

$$\Phi(x, y) = \frac{1}{(2\pi)^{2d}} \int e^{-ipx - iqy} \Psi(p, q) \mathrm{d}p\, \mathrm{d}q \in S_{1-\alpha}^{\alpha}(\mathbb{R}^{2d}).$$

In particular, Φ satisfies the estimate

$$|\Phi(x, y)| \le C \exp\{-|x/A|^{1/(1-\alpha)} - |y/A|^{1/(1-\alpha)}\}$$

with some $A > 0$. Therefore the convolution $\Phi * f$ exists for any function $f \in L^2(\mathbb{C}^d, e^{-2\rho}\mathrm{d}\lambda)$ if $\alpha > \beta$. Let $\rho_1 = \rho_{U,B_1,N}$, where $B_1 > B$. Applying the triangle inequality to each term of ρ, we find that

$$\int |\Phi(x', y')|^2 e^{2\rho(x-x', y-y')} \mathrm{d}\lambda' \le C' e^{2\rho_1(x,y)}. \tag{35}$$

Using next the Cauchy-Schwarz-Bunyakovskii inequality, we obtain

$$|(\Phi * f)(x, y)| \le C' \|f\|_{U,B,N} e^{\rho_1(x,y)}.$$

Choosing $B_1 < B'$, we see that the correspondence $f \to \Phi * f$ is a continuous map from $L^2(\mathbb{C}^d, e^{-2\rho}\mathrm{d}\lambda)$ to $L^2(\mathbb{C}^d, e^{-2\rho'}\mathrm{d}\lambda)$. Moreover, it is a Hilbert-Schmidt map. Indeed, the multiplication by $e^{-\rho}$ is an isometry from $L^2(\mathbb{C}^d, e^{-2\rho}\mathrm{d}\lambda)$ onto $L^2(\mathbb{C}^d)$ and the map in question belongs to the Hilbert-Schmidt class if and only if the integral operator on $L^2(\mathbb{C}^d)$ with kernel $e^{-\rho'(x',y')}\Phi(x - x', y - y')e^{\rho(x,y)}$ is in the same class, that is, if the kernel is square-integrable, and this is ensured by the estimate (35).

On the other hand, the map $f \to \Phi * f$ is identified with the infinite-order differential operator $\psi(-\Delta) = 1 + \sum_{k \ge 1} c_k \Delta^k$ if f is treated as a generalized function defined on appropriate test functions. According to [1], such an operator is well defined on any space $S_{1-\beta'}^{\beta'}(\mathbb{R}^{2d})$, where $\beta' < \alpha$. If $\beta' > \beta$, then all elements of $L^2(\mathbb{C}^d, e^{-2\rho}\mathrm{d}\lambda)$ are integrable with test functions in this space. With this choice of β', we have the chain of identities

$$(f, \psi(-\Delta)\varphi) = \lim_{n \to \infty} \left(f, \left(1 + \sum_{k=1}^{n} c_k \Delta^k \right) \varphi \right) = (2\pi)^{-2d} (\tilde{f}, \Psi \tilde{\varphi}) = (f, \Phi * \varphi),$$

where φ is any element of $S^{\beta'}_{1-\beta'}(\mathbb{R}^{2d})$. In particular, $(\Phi * f, \varphi) = (f, \varphi)$ for all $f \in H^{\beta,B}_N(U)$. It follows that $\Phi * f = f$ because $S^{\beta'}_{1-\beta'}(\mathbb{R}^{2d})$ has a sufficiently large stock of functions (see [1]). This proves Lemma 2. □

Theorem 5. *The spaces $S^{\beta,b}(U)$ and $S^{\beta}(U)$ are nuclear for any open cone $U \subset \mathbb{R}^d$. The spaces $S^{\beta}(K)$ associated with closed cones are also nuclear.*

Proof. The statement for $S^{\beta,b}(U)$ follows immediately from Lemma 2 because the composite of two Hilbert-Schmidt maps is nuclear and the projective limit of a sequence of Hilbert spaces with nuclear connecting maps is a nuclear Fréchet space. The statement about $S^{\beta}(U)$ and $S^{\beta}(K)$ follows from the heredity properties of inductive limits of countable families of nuclear spaces (see [11]). □

Corollary 2. *The spaces $S^{\beta,b}(U)$ and $S^{\beta}(U)$ are reflexive. Moreover, they are Montel spaces.*

Indeed, they are complete and barrelled, and every nuclear space with these properties is a Montel space (see [11], Ch. IV, Exercise 19). It is still an open question whether the spaces $S^{\beta}(K)$ over closed cones have these properties. But their completions certainly have them.

8. Kernel theorems

If E_1 and E_2 are locally convex spaces (LCS), then their (algebraic) tensor product equipped with the projective topology τ_π is denoted by $E_1 \otimes_\pi E_2$, and the same product with the inductive topology τ_ι is denoted by $E_1 \otimes_\iota E_2$. If E_1 and E_2 are Hilbert spaces, then we write $E_1 \otimes_{\mathrm{H}} E_2$ for their tensor product equipped with the natural inner product. The completion of each of these spaces is denoted by a "hat" over the tensor product symbol.

Lemma 3. *Let U_i be open cones in $\mathbb{R}^{d_i}$, $i = 1, 2$. Then there is a canonical isomorphism*

$$H^{\beta,B}_N(U_1) \hat{\otimes}_{\mathrm{H}} H^{\beta,B}_N(U_2) \simeq H^{\beta,B}_N(U_1 \times U_2),$$

defined by identifying $f_1 \otimes f_2$ with the function $f_1(z_1)f_2(z_2)$.

Proof. We use the same line of reasoning as in the case of square-integrable functions. (This case is considered, for example, in [15].) By the property (4), all function of the form $f_1(z_1)f_2(z_2)$, where $f_1 \in H^{\beta,B}_N(U_1)$ and $f_2 \in H^{\beta,B}_N(U_2)$, belong to the space $H^{\beta,B}_N(U_1 \times U_2)$, and their linear span is identified with $H^{\beta,B}_N(U_1) \otimes H^{\beta,B}_N(U_2)$. The natural inner product on a tensor product of Hilbert spaces is defined by

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle = \langle f_1, g_1 \rangle \langle f_2, g_2 \rangle$$

with subsequent extension by linearity. In our case it obviously coincides with the inner product induced by that of $H^{\beta,B}_N(U_1 \times U_2)$. If $\{f_j\}$ and $\{g_k\}$ are bases in the spaces whose tensor product is being formed, then $\{f_j(z_1)g_k(z_2)\}$ is an orthonormal system in $H^{\beta,B}_N(U_1 \times U_2)$ and Fubini's theorem immediately shows that this system is total. Therefore the completion of the tensor product coincides with $H^{\beta,B}_N(U_1 \times U_2)$. □

Lemma 4. *Let $h_1 \colon E_1 \to F_1$ and $h_2 \colon E_2 \to F_2$ be Hilbert-Schmidt maps between Hilbert spaces. Then the map*

$$E_1 \otimes_{\mathrm{H}} E_2 \xrightarrow{h_1 \otimes h_2} F_1 \otimes_\pi F_2$$

is continuous.

Proof. We assume that all the spaces are separable because this is the case in the applications below, although this lemma holds in the general case as well. A map $h\colon E \to F$ belongs to the Hilbert-Schmidt class if and only if

$$\|h\|_2 \stackrel{\text{def}}{=} \left(\sum_j \|he_j\|^2\right)^{1/2} < +\infty$$

for some (and thus for every) orthonormal basis $\{e_j\}$ in E. According to [11], Ch. III, § 6.3, the projective topology on $F_1 \otimes F_2$ is determined by the tensor product of the norms on F_1 and F_2. Denoting this product by $\|\cdot\|_\pi$, we recall that it is a cross-norm, that is, $\|f_1 \otimes f_2\|_\pi = \|f_1\| \cdot \|f_2\|$. Moreover, it is stronger than any other cross-norm. In particular, it is stronger than the Hilbert norm determined by the inner product. Let $\{e^1_j\}$ and $\{e^2_k\}$ be orthonormal bases in E_1 and E_2 respectively. Then $\{e^1_j \otimes e^2_k\}$ is an orthonormal basis in $E_1 \hat{\otimes}_{\mathrm{H}} E_2$ and every element g of this space can be written as $g = \sum \lambda_{jk}\, e^1_j \otimes e^2_k$. Using the cross-property of $\|\cdot\|_\pi$, the Cauchy-Schwarz-Bunyakovskii inequality and Parseval's identity $\|g\|^2 = \sum |\lambda_{jk}|^2$, we obtain

$$\|(h_1 \otimes h_2)\left(\sum_{jk \le n} \lambda_{jk}\, e^1_j \otimes e^2_k\right)\|_\pi \le \sum_{j,k \le n} |\lambda_{jk}|\, \|h_1(e^1_j)\|\, \|h_2(e^2_k)\| \le \|g\|\, \|h_1\|_2\, \|h_2\|_2.$$

It follows that the family $\lambda_{jk}\, h_1(e^1_j) \otimes h_2(e^2_k)$ of elements of the Banach space $F_1 \hat{\otimes}_\pi F_2$ is absolutely summable. Hence we have defined a continuous map $E_1 \hat{\otimes}_{\mathrm{H}} E_2 \to F_1 \hat{\otimes}_\pi F_2$. This map coincides with $h_1 \otimes h_2$ on the basis elements and hence on all elements of $E_1 \otimes E_2$ because the canonical bilinear map of $F_1 \times F_2$ to $F_1 \hat{\otimes}_{\mathrm{H}} F_2$ is continuous. The lemma is proved. $\square$

Lemma 5. *The space $S^{\beta,b}(U_1) \otimes S^{\beta,b}(U_2)$ is dense in $S^{\beta,b}(U_1 \times U_2)$ for every pair of open cones $U_i \subset \mathbb{R}^{d_i}$, $i = 1, 2$, and any $0 \le \beta < 1$, $b > 0$.*

Proof. Let $1 < \alpha < 2$ and let g be an entire function on $\mathbb{C}^{d_1+d_2}$ satisfying

$$|g(z)| \le C \exp\left\{-\sum_j |x_j|^{1/\alpha} + |by|^{1/(1-\beta)}\right\}$$

and such that $g(0) = 1$. We take $f \in S^{\beta,b}(U_1 \times U_2)$ and consider the sequence $f_\nu(z) = f((1 - 1/\nu)z)g(z/(2\nu))$, $\nu = 1, 2, \ldots$. Setting $\epsilon = 1/\nu$, $p = 1/(1-\beta)$, and using the inequalities $1 \ge (1-\epsilon)^p + \epsilon^p > (1-\epsilon)^p + (\epsilon/2)^p$, we easily verify that f_ν is bounded in each of the norms of $S^{\beta,b}(U_1 \times U_2)$. Therefore $f_\nu \to f$ in the topology of this space because it is a Montel space and its topology is stronger than the topology of pointwise convergence. Let $H_{1/2}(U)$ denote the Hilbert space of entire functions belonging to $L^2(\mathbb{C}^d, e^{-2\rho_{1/2}}\mathrm{d}\lambda)$, where

$$\rho_{1/2} = \exp\left\{-\sum_j |x_j|^{1/2} + d(bx, U)^{1/(1-\beta)} + |by|^{1/(1-\beta)}\right\}.$$

Clearly, it is continuously embedded in $S^{\beta,b}(U)$. All the functions f_ν are contained in $H_{1/2}(U_1 \times U_2)$. Indeed, if $B > b$ and is sufficiently close to b, then

$$\|f_\nu\|_{1/2} \le C_\nu \|f\|_{U_1 \times U_2, B, 0}.$$

The arguments used in the proof of Lemma 3 show that $H_{1/2}(U_1) \otimes H_{1/2}(U_2)$ is dense in $H_{1/2}(U_1 \times U_2)$. Therefore every function f_ν can be approximated by elements of the tensor product in a metric stronger than that of $S^{\beta,b}(U_1 \times U_2)$. This proves the lemma. $\square$

Theorem 6. *Let U_1, U_2 be open cones in $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$. Then there are the canonical isomorphisms*

$$S^{\beta,b}(U_1)\,\hat{\otimes}_\iota\, S^{\beta,b}(U_2) \simeq S^{\beta,b}(U_1\times U_2) \tag{36}$$

$$S^{\beta}(U_1)\,\hat{\otimes}_\iota\, S^{\beta}(U_2) \simeq S^{\beta}(U_1\times U_2). \tag{37}$$

Proof. The topologies τ_ι and τ_π coincide on tensor products of Fréchet spaces (see [11], Ch. III, § 6.5), and Lemmas 2–4 show that the topology τ_π on $S^{\beta,b}(U_1)\otimes S^{\beta,b}(U_2)$ coincides with the topology induced by that of $S^{\beta,b}(U_1\times U_2)$ because the systems of norms determining these topologies are equivalent. By Lemma 5, the natural injection $S^{\beta,b}(U_1)\otimes_\iota S^{\beta,b}(U_2)\to S^{\beta,b}(U_1\times U_2)$ has a unique extension to the completion of the tensor product and this extension is an isomorphism. This proves (36). Isomorphism (37) follows from (36) because of two facts. First, if E_ν and F_ν are injective sequences of locally convex spaces and their inductive limits are Hausdorff spaces[4] then

$$(\varinjlim E_\nu)\otimes_\iota(\varinjlim F_\nu) = \varinjlim(E_\nu\otimes_\iota F_\nu). \tag{38}$$

Second, if G_ν is an injective sequence of locally convex spaces and the limit $\varinjlim \hat{G}_\nu$ of their completions is a Hausdorff space, then

$$\widehat{\varinjlim G_\nu} = \widehat{\varinjlim \hat{G}_\nu}. \tag{39}$$

We set $E_\nu = S^{\beta,\nu}(U_1)$, $F_\nu = S^{\beta,\nu}(U_2)$, $G_\nu = E_\nu\otimes_\iota F_\nu$, and successively use (38), (39), (36). Since $S^\beta(U_1\times U_2)$ is complete, we get (37).

The relation (38) is actually a part of Proposition 14 in Ch. I of [9]. We note that the topology τ_ι on a tensor product of locally convex spaces is certainly Hausdorff because it is stronger than the topology τ_π, which is Hausdorff by [10], Ch. VII, Proposition 8. The proof of (38) consists of using the definition of τ_ι as the topology of uniform convergence on separately equicontinuous sets of bilinear forms and noting that a set of bilinear forms on $(\varinjlim E_\nu)\times(\varinjlim F_\nu)$ is separately equicontinuous if and only if the same is true for the sets of their restrictions to each of $E_\nu\times F_\nu$. To prove (39), we start by noting that the continuous injections $u_{\mu\nu}\colon G_\mu\to G_\nu$, $\nu>\mu$, generate continuous maps $\hat{u}_{\mu\nu}\colon \hat{G}_\mu\to\hat{G}_\nu$ which still satisfy the chain rule $\hat{u}_{\nu\mu}\circ\hat{u}_{\mu\lambda}=\hat{u}_{\nu\lambda}$. Therefore the space $\varinjlim\hat{G}_\nu$ is well defined. For every ν, we have the continuous map $G_\nu\to\varinjlim\hat{G}_\nu$. It is injective because the restriction of $\hat{u}_{\mu\nu}$ to G_ν is one-to-one whenever $\mu>\nu$. These injections determine a continuous injection

$$\varinjlim G_\nu \to \varinjlim \hat{G}_\nu. \tag{40}$$

In particular, if $\varinjlim\hat{G}_\nu$ is Hausdorff, then so is $\varinjlim G_\nu$. On the other hand, the injections $G_\nu\to\varinjlim G_\nu$ extend to maps $\hat{G}_\nu\to\widehat{\varinjlim G_\nu}$ and generate a continuous map

$$\varinjlim\hat{G}_\nu\to\widehat{\varinjlim G_\nu}.$$

The composite of this map and (40) is the canonical embedding of the space $\varinjlim G_\nu$ into its completion. Therefore the topology of $\varinjlim G_\nu$ coincides with the topology induced by that of $\varinjlim\hat{G}_\nu$. Since the image of $\varinjlim G_\nu$ is dense in $\varinjlim\hat{G}_\nu$, the injection (40) extends to an isomorphism, which completes the proof of Theorem 6. □

To every closed cone in $\mathbb{R}^{d_1+d_2}$ having the product structure $K_1\times K_2$ with $K_i\subset\mathbb{R}^{d_i}$ we assign the space

$$S^\beta(K_1,K_2) = \varinjlim_{U_1,U_2} S^\beta(U_1\times U_2), \tag{41}$$

where the U_i are open cones in $\mathbb{R}^{d_i}$ such that $U_i\Supset K_i$, $i=1,2$.

[4]The very definition of a LCS usually requires the space to be Hausdorff. But this property can be lost after taking an inductive limit.

The existence of canonical embeddings $S^\beta(U_1 \times U_2) \to S^\beta(\{0\})$ enables us to interpret (41) as an inductive limit in the sense of the definition given in [9, 10]. However, if one (or both) of the cones K_i is degenerate, then the set of open cones involved in the limit is not directed. Directed sets are sometimes more convenient to use, and the situation can easily be remedied by taking a limit over all cones with open projections in $\mathbb{R}^{d_i}$ instead of open cones. In other words, we can simply add the cone $\{0\}$ to the set of open cones in each $\mathbb{R}^{d_i}$. This agrees with what was said in Sect. 2. Spaces associated with relatively open cones $U_1 \times \{0\}$ and $\{0\} \times U_2$ are defined in the same manner as those associated with open cones, and Theorem 6 (as well as Theorem 1) can be immediately extended to them. In particular, if $\{0\}$ is a degenerate cone in $\mathbb{R}^{d_1}$, then

$$S^\beta(\{0\}) \hat{\otimes}_\iota S^\beta(U_2) \simeq S^\beta(\{0\} \times U_2).$$

Such a unification was used in [16], where an analogue of (41) was proposed for the DFS-spaces S^β_α. We emphasize that the addition of $\{0\}$ to the set of open cones leaves inductive limit (41) unchanged. For instance, let $K_1 = \{0\}$. Choose any two open cones in $\mathbb{R}^{d_1}$ with disjoint closures of their projections (say, the positive and negative orthants U_+ and U_-). Theorem 3 shows that every element of $S^\beta(\{0\} \times U_2)$ can be written as a sum of functions belonging to $S^\beta(U_+ \times V_2)$ and $S^\beta(U_- \times V_2)$, where $U_2 \Supset V_2 \Supset K_2$. Using the open mapping theorem, we see that the inductive topology on $S^\beta(\{0\}, K_2)$ with respect to the family of subspaces $S^\beta(U_1 \times U_2)$, where $U_i \Supset K_i$, coincides with that determined by the subfamily $S^\beta(U_\pm \times U_2)$, where $U_2 \Supset K_2$. It also coincides with the inductive limit topology with respect to the increasing family $S^\beta(\{0\} \times U_2)$.

Theorem 7. *Let K_i be a closed cone in $\mathbb{R}^{d_i}$, $i = 1, 2$. Every separately continuous bilinear form w on $S^\beta(K_1) \times S^\beta(K_2)$ is uniquely representable as*

$$w(f, g) = (v, f \otimes g),$$

where v is a continuous linear functional on $S^\beta(K_1, K_2)$.

Proof. By the isomorphism (37), the restriction of w to $S^\beta(U_1) \times S^\beta(U_2)$, where $K_i \Subset U_i$, uniquely determines a continuous linear functional on $S^\beta(U_1 \times U_2)$. If both the cones K_1, K_2 are nondegenerate, then the family of neighborhoods $U_1 \times U_2$ is decreasing and hence we have defined a linear functional on $S^\beta(K_1, K_2)$, which is continuous by the definition of the inductive topology. The same argument works in the degenerate case if we use the above-stated unification of definition (41) and the corresponding generalization of Theorem 6. Another way is to use Theorem 3. For instance, let $K_1 = \{0\}$ as above. If $v_\pm$ are the functionals generated by w on $S^\beta(U_\pm \times V_2)$, then we can use the decomposition $f = f_+ + f_-$ with $f_\pm \in S^\beta(U_\pm \times V_2)$ and define v by $v(f) = v_+(f_+) + v_-(f_-)$. This functional is well defined because v_+ and v_- coincide on $S^\beta(U_+ \times V_2) \cap S^\beta(U_- \times V_2) = S^\beta((U_+ \cup U_-) \times V_2)$ by (37). $\square$

Corollary 3. *The space of all separately continuous bilinear forms on $S^\beta(K_1) \times S^\beta(K_2)$ can be identified with the space $S'^\beta(K_1, K_2)$, which is the dual of $S^\beta(K_1, K_2)$.*

9. Carrier cones of multilinear forms

Now we turn to the carrier cones of functionals generated by multilinear forms on $S^\beta(\mathbb{R}^{d_1}) \times \cdots \times S^\beta(\mathbb{R}^{d_n})$ with given carrier cones for every argument. Our main concern is about the extension of such forms to larger spaces. The general theory of extension of multilinear forms is based on the notion of hypocontinuity [11], but we need only the following simple lemma.

Lemma 6. *Let L be a sequentially dense subspace of a locally convex space E_1, and let E_2 be a barrelled space. Then every separately continuous form bilinear w on $L \times E_2$ has a unique extension to $E_1 \times E_2$ which is bilinear and separately continuous.*

Proof. For every fixed $g \in E_2$, the form $w(f, g)$ extends uniquely to E_1 by continuity. We denote this extension by $\hat{w}$. We must verify that this functional is linear and continuous in g for every fixed $f \in E_1$. Choose a sequence $f_\nu \in L$ that converges to f and denote the corresponding elements of E_2' by $\mathbf{f}_\nu$. Then $\mathbf{f}_\nu(g) = w(f_\nu, g)$. It is well known (see [10] or [11]) that if E_2 is barrelled, then the pointwise convergence of the sequence $\mathbf{f}_\nu \in E_2'$ implies that its limit $\mathbf{f}$ belongs to E_2', i.e., is linear and continuous. We have $\mathbf{f}(g) = \hat{w}(f, g)$. This proves the lemma. □

Definition 2. Let w be a multilinear separately continuous form on $S^\beta(\mathbb{R}^{d_1}) \times \cdots \times S^\beta(\mathbb{R}^{d_n})$, and let $K = K_1 \times \cdots \times K_n$, where K_j is a closed cones in $\mathbb{R}^{d_j}$, $j = 1, \dots, n$. We say that K is a *carrier cone* of w, if every K_j is a carrier cone for all linear functionals defined on $S^\beta(\mathbb{R}^{d_j})$ by $w(f_1, \dots, f_n)$ with fixed $f_i \in S^0(\mathbb{R}^{d_i})$, $i \neq j$.

Theorem 8. *Let $K = K_1 \times \cdots \times K_n$, where $K_j \subset \mathbb{R}^{d_j}$. If K is a carrier cone of a multilinear separately continuous form w on $S^\beta(\mathbb{R}^{d_1}) \times \cdots \times S^\beta(\mathbb{R}^{d_n})$, then K is also a carrier cone of the functional $v \in S'^\beta(\mathbb{R}^{d_1+\cdots+d_n})$ generated by this form.*

Proof. For simplicity we assume that $d_j = d$ for all j. It suffices to show that every cone $\mathbb{R}^{d(j-1)} \times K_j \times \mathbb{R}^{d(n-j)}$ is a carrier of v (so that their intersection is also a carrier by Theorem 4). We put $j = 1$ without loss of generality. Suppose that $n = 2$ and U_1 is an open cone in $\mathbb{R}^d$ such that $K_1 \Subset U_1$. By Theorem 1, each element $f \in S^\beta(U_1)$ can be approximated by elements of $S^\beta(\mathbb{R}^d)$ in the metric of some $S^{\beta,b}(U_1)$ (where b is dependent on f). Applying Lemma 6, we extend w to a bilinear separately continuous form on $S^\beta(U_1) \times S^\beta(\mathbb{R}^d)$. By Theorem 6, this form in turn determines a linear continuous functional on $S^\beta(U_1 \times \mathbb{R}^d)$, which is an extension of v because $S^\beta(\mathbb{R}^d) \otimes S^\beta(\mathbb{R}^d)$ is dense in $S^\beta(\mathbb{R}^{2d})$. This proves Theorem 8 for bilinear forms because the intersection of all cones $\bar{U}_1 \times \mathbb{R}^d$ is equal to $K_1 \times \mathbb{R}^d$.

Now we use induction on n. We regard the n-linear form w ($n > 2$) as a bilinear form on $L \times E_2$, where $L = \overset{n-1}{\otimes} S^\beta(\mathbb{R}^d)$ and $E_2 = S^\beta(\mathbb{R}^d)$. By the inductive hypothesis, this form is separately continuous if L is given the topology induced by that of $E_1 = S^\beta(U_1 \times \mathbb{R}^{d(n-1)})$. The subspace L is sequentially dense in E_1 because every element of $S^\beta(U_1 \times \mathbb{R}^{d(n-1)})$ can be approximated in the metric of $S^{\beta,b}(U_1 \times \mathbb{R}^{d(n-1)})$ by elements of some $S^{\beta,b}(\mathbb{R}^{dn})$, which can in turn be approximated by elements of $\overset{n-1}{\otimes} S^{\beta,b}(\mathbb{R}^d)$ in the (stronger) metric of $S^{\beta,b}(\mathbb{R}^{dn})$. Again using Lemma 6, we conclude that v has a continuous extension to $S^\beta(U_1 \times \mathbb{R}^{d(n-1)})$. This proves the theorem. □

Theorems 7 and 8 raise the question of the relation between the spaces $S^\beta(K_1 \times K_2)$ and $S^\beta(K_1, K_2)$. Clearly, we have

$$S^\beta(K_1 \times K_2) \subset S^\beta(K_1, K_2)$$

because $U \Supset K_1 \times K_2$ implies that $U \supset U_1 \times U_2$, where $U_i = U \cap \mathbb{R}^{d_i}$ if $K_i \neq \{0\}$ and $U_i = \{0\}$ otherwise. As a rule, this inclusion is strict.

Theorem 9. *The spaces $S^\beta(K_1 \times K_2)$ and $S^\beta(K_1, K_2)$ coincide only if $K_1 = \mathbb{R}^{d_1}$ and $K_2 = \mathbb{R}^{d_2}$ or if both these cones are degenerate. In all other cases, these spaces are distinct and have different dual spaces.*

A proof of Theorem 9 is given in Appendix 2. It gives an idea of the stock of functions in the space S^β that ensures the angular localizability of the functionals belonging to its dual. An obvious generalization of definition (41) is

$$S^\beta(K_1, \dots, K_n) = \varinjlim_{U_1, \dots U_n} S^\beta(U_1 \times \cdots \times U_n), \tag{42}$$

where $K_j \Subset U_j$. If a functional $v \in S'^{\beta}(\mathbb{R}^{d_1+\dots+d_n})$ admits a continuous extension to the space (42), then we say that it is *strongly carried* by the cone $K_1 \times \dots \times K_n$. These are precisely those functionals that are generated by multilinear forms carried by $K_1 \times \dots \times K_n$. To prove this, we need another decomposition theorem.

Theorem 10. *Let K_j be a closed cone in $\mathbb{R}^{d_j}$, $j = 1, \dots, n$. Every $f \in S^{\beta}(K_1, \dots, K_n)$ can be decomposed as $f = f_1 + \dots + f_n$, where $f_j \in S^{\beta}(\mathbb{R}^{d_1+\dots+d_{j-1}}, K_j, \mathbb{R}^{d_{j+1}\dots+d_n})$.*

Proof. This is basically the same as that of Lemma 3 in [16] on the DFS spaces $S^{\beta}_{\alpha}(K_1, \dots, K_n)$. Suppose that $n = 2$. If K_1 and K_2 are nondegenerate, then there are open cones V_1 and V_2 such that $f \in S^{\beta}(V_1 \times V_2)$. In the degenerate case, f is a sum of elements of such spaces. It suffices to consider the first case. We choose open cones V'_j such that $V_j \Supset V'_j \Supset K_j$ and use the notation $U = V_1 \times V_2$, $U_1 = V'_1 \times \complement\bar{V}_2$, $U_2 = \complement\bar{V}_1 \times V'_2$. Then $\bar{U}_1 \cap \bar{U}_2 = \{0\}$ and it follows from Theorem 3 that $f = f_1 + f_2$, where $f_{1,2} \in S^{\beta}(U \cup U_{1,2})$. Furthermore, $\overline{U \cup U_1} = (\bar{V}_1 \times \bar{V}_2) \cup \bar{U}_1 \supset V'_1 \times \mathbb{R}^{d_2}$. Since the inclusion $\overline{W} \supset W'$ implies that $S^{\beta}(W) \subset S^{\beta}(W')$, we conclude that $f_1 \in S^{\beta}(V'_1 \times \mathbb{R}^{d_2}) \subset S^{\beta}(K_1, \mathbb{R}^{d_2})$. Similarly, $f_2 \in S^{\beta}(\mathbb{R}^{d_1}, K_2)$. The same argument shows that every element of $S^{\beta}(V_1 \times \dots \times V_n)$ (with m cones $V_j \Supset K_j$ being different from $\mathbb{R}^{d_j}$) is representable as a sum of two functions belonging to the spaces $S^{\beta}(V'_1 \times \dots \times V'_n)$, where $V_j \Supset V'_j \Supset K_j$ and $m-1$ cones V'_j are different from $\mathbb{R}^{d_j}$. Indeed, let the cones $V_j \neq \mathbb{R}^{d_j}$ occupy the first place. For the rest, we set $V'_j = V_j = \mathbb{R}^{d_j}$ and now use the notation $U = V_1 \times \dots \times V_n$, $U_1 = V'_1 \times \complement\bar{V}_2 \times V'_3 \times \dots \times V'_n$, $U_2 = \complement\bar{V}_1 \times V'_2 \times V'_3 \times \dots \times V'_n$. Then $\bar{U}_1 \cap \bar{U}_2 \Subset U$ and Theorem 3 again applies. □

Theorem 11. *Let K_j be a closed cone in $\mathbb{R}^{d_j}$, $j = 1, \dots n$. A functional $v \in S'^{\beta}(\mathbb{R}^{d_1+\dots+d_n})$ is strongly carried by the cone $K_1 \times \dots \times K_n$ if and only if v is generated by a multilinear separately continuous form on $S^{\beta}(\mathbb{R}^{d_1}) \times \dots \times S^{\beta}(\mathbb{R}^{d_n})$ that is carried by this cone.*

Proof. Let $v \in S'^{\beta}(K_1, \dots, K_n)$, $U_j \Supset K_j$ and $f_j \in S^{\beta}(U_j)$. It is clear from (4) that the map

$$f_j \to f_1 \otimes \dots \otimes f_j \otimes \dots \otimes f_n$$

from $S^{\beta}(U_j)$ to $S^{\beta}(\mathbb{R}^{d(j-1)} \times U_j \times \mathbb{R}^{d(n-j)})$ is continuous. Therefore, the multilinear form corresponding to v is certainly carried by $K_1 \times \dots \times K_n$. To prove the converse, we again set $d_j = d$ for simplicity. If some of the K_j are degenerate, then it is convenient to use the unification of definitions (41), (42) mentioned in Sect. 8. In the proof of Theorem 8 we saw that $v \in S'^{\beta}(\mathbb{R}^{d(j-1)}, K_j, \mathbb{R}^{d(n-j)})$ for any j. Let us show that there is a continuous extension $\hat{v}$ to the space $E = S^{\beta}(K_1, K_2, \mathbb{R}^{d(n-2)})$. By Theorem 10, this space is the sum of the two subspaces $L_1 = S^{\beta}(K_1, \mathbb{R}^{d(n-1)})$ and $L_2 = S^{\beta}(\mathbb{R}^d, K_2, \mathbb{R}^{d(n-2)})$. If $f = f_1 + f_2$, we set $\hat{v}(f) = v_1(f_1) + v_2(f_2)$. This extension is well defined because v_1 and v_2 coincide on $L_1 \cap L_2$. Indeed, this intersection is the inductive limit of the increasing family of spaces $S^{\beta}(U)$, where U is the union of cones $U_1 \times \mathbb{R}^{d(n-1)}$ and $\mathbb{R}^d \times U_2 \times \mathbb{R}^{d(n-2)}$ with $U_1 \Supset K_1$ and $U_2 \Supset K_2$. By Theorem 1, the space $S^{\beta}(\mathbb{R}^{dn})$ is dense in this intersection, whose topology is stronger than the topologies of L_1 and L_2. The functional $\hat{v}$ is obviously continuous in the inductive topology determined by the injections $L_i \to E$, $i = 1, 2$, and this topology coincides with the original topology of E by the open mapping theorem. Applying the same arguments to the triple $E = S^{\beta}(K_1, K_2, K_3, \mathbb{R}^{d(n-3)})$, $L_1 = S^{\beta}(K_1, K_2, \mathbb{R}^{d(n-2)})$, $L_2 = S^{\beta}(\mathbb{R}^{2d}, K_3, \mathbb{R}^{d(n-3)})$ and so on, we complete the proof after finitely many steps. □

10. A Paley-Wiener-Schwartz-type theorem

Let V be an open cone in $\mathbb{R}^d$ and let $V^* = \{x : x\eta \geq 0, \forall \eta \in V\}$ be its dual cone. As shown in [12], the Laplace transformation maps the space $S'^{\beta}(V^*)$, $\beta > 0$, onto an algebra of analytic functions defined on the tubular domain $T^V = \mathbb{R}^{dn} + iV$ and satisfying certain bounds on their growth near the real boundary of the domain of analyticity and at infinity.

An analogous theorem was proved in [4] for the class S'^0, which requires more sophisticated reasoning. Theorem 11 enables us to extend these results to the spaces $S'^{\beta}(V_1^*,\dots,V_n^*)$.

Let $\beta>0$, let V_j be open cones in $\mathbb{R}^{d_j}$, $j=1,\dots,n$, and let $V=V_1\times\cdots\times V_n$. We denote by $\mathcal{A}_\beta(V_1,\dots,V_n)$ the space of functions analytic in the domain $T^V=\mathbb{R}^{dn}+iV$ and satisfying the condition

$$|\mathbf{u}(\zeta)|\le C_{\epsilon,W_1,\dots,W_n}\prod_{j=1}^{n}|\operatorname{Im}\zeta_j|^{-N}\exp\{\epsilon|\operatorname{Re}\zeta_j|^{1/\beta}\},\qquad \operatorname{Im}\zeta_j\in W_j,\ j=1,\dots,n \tag{43}$$

for any $\epsilon>0$, any cones $W_j\Subset V_j$ and some N depending on ϵ and these cones. If $\beta=0$, then we define $\mathcal{A}_0(V_1,\dots,V_n)$ as the space of functions analytic on the same domain and satisfying

$$|\mathbf{u}(\zeta)|\le C_{R,W_1,\dots,W_n}\prod_{j=1}^{n}|\operatorname{Im}\zeta_j|^{-N}\qquad \operatorname{Im}\zeta_j\in W_j,\quad |\zeta_j|\le R,\quad j=1,\dots,n, \tag{44}$$

where N depends on $R>0$ and on W_j. Clearly, these spaces are algebras under pointwise multiplication.

Theorem 12. *The Laplace transformation $\mathcal{L}: v\to(v,e^{iz\zeta})$ is an isomorphism of the space $S'^{\beta}(V_1^*,\dots,V_n^*)$, $0\le\beta<1$, onto the algebra $\mathcal{A}_\beta(V_1,\dots,V_n)$. The analytic function $(\mathcal{L}v)(\zeta)$ tends to the Fourier transform $\tilde v$ of v in the strong topology of $S'_\beta(\mathbb{R}^{dn})$ as $\operatorname{Im}\zeta\to 0$ inside a fixed cone $W_1\times\cdots\times W_n$, where $W_j\Subset V_j$.*

Proof. Since $S'^0(V_1^*,\dots,V_n^*)\subset S'^0(V^*)$, we can use Theorem 4 in [12] for $\beta>0$ and Theorem 2 in [4] for $\beta=0$. Their statements are identical to that of Theorem 12 for $n=1$. In particular, they show that every functional belonging to $S'^{\beta}(V^*)$ has a Laplace transform, which is analytic in T^V and whose boundary value is $\tilde v$. The bounds (43) and (44) are stronger than the bounds in [12, 4], which hold for an arbitrary element of $S'^{\beta}(V^*)$. However, the multiplicative property (4) enables us to derive them in the same way, starting from the estimate

$$|\mathcal{L}v(\zeta)|=|(v,e^{iz\zeta})|\le\|v\|'_{U,B,N}\|e^{iz\zeta}\|'_{U,B,N}, \tag{45}$$

where we use the norms (5) and their dual norms. Here B can be taken arbitrarily large, $U=U_1\times\cdots\times U_n$, where U_j are any cones with open projections such that $V_j^*\Subset U_j$, and N generally depends on B and U. We choose the cones U_j and auxiliary cones U'_j so that $V_j^*\Subset U_j\Subset U'_j\Subset\operatorname{Int}W_j^*$, where $\operatorname{Int}W_j^*$ is the interior of W_j^*. This is possible because $W_j\Subset V_j$ implies that $V_j^*\Subset\operatorname{Int}W_j^*$. Let $\beta=0$ and $\zeta=\xi+i\eta$. Then

$$\|e^{iz\zeta}\|'_{U,B,N}=\sup_{x,y}\exp\{-x\eta-y\xi+N\ln(1+|x|)-Bd(x,U)-B|y|\}. \tag{46}$$

This exponential is factorizable, and each factor can be estimated in the same manner. Namely, assuming that $|\xi_j|\le R<B$, we can omit terms that depend on y_j. If $x_j\notin U'_j$, then $d(x_j,U_j)>\theta|x_j|$ with some $\theta>0$, and the expression in the exponent is dominated by a constant for $|\eta_j|\le R<\theta B$,. If $x_j\in U'_j$, then the inclusion $U'_j\Subset\operatorname{Int}W_j^*$ implies that there is a $\theta'>0$ such that $x_j\eta_j\ge\theta'|x_j||\eta_j|$ for all $x_j\in U'_j$ and $\eta_j\in W_j$. Substituting this inequality in (46), dropping the negligible term $d(x_j,U_j)$, and locating the extremum, we obtain (44) with some constant $C_{R,W_1,\dots,W_n}$ proportional to $\|v\|'_{U,B,N}$. The case $\beta>0$ is treated in the same way, with obvious changes in computation.

The nontrivial part of Theorem 12 states that any function belonging to the algebra $\mathcal{A}_\beta(V_1,\dots,V_n)$ is the Laplace transform of an element in $S'^{\beta}(V_1^*,\dots,V_n^*)$. Let $\mathbf{u}$ be a function with property (44) and let u be its boundary value, which exists in the Schwartz space $\mathcal{D}'(\mathbb{R}^{dn})=S'_0(\mathbb{R}^{dn})$ of distributions by Theorem 3.1.15 of [6]. By Theorem 4 in [12], the stronger condition (43) implies that the distribution u belongs to $S'_\beta(\mathbb{R}^{dn})$. Restricting

it to test functions of the form $g_1 \otimes \cdots \otimes g_n$, where $g_j \in S_\beta(\mathbb{R}^d)$, and using the same theorem for $\beta > 0$ and Theorem 2 in [4] for $\beta = 0$, we conclude that the multilinear form determined by the inverse Fourier transform of u is carried by the cone $V_1^* \times \cdots \times V_n^*$. An application of Theorem 11 completes the proof. $\square$

We also note that every cone V has the same dual cone as its convex hull $\operatorname{ch} V$. Hence Theorem 12 implies that $\mathcal{A}_\beta(V_1, \ldots, V_2) = \mathcal{A}_\beta(\operatorname{ch} V_1, \ldots, \operatorname{ch} V_n)$.

Appendix 1.

Proof of Lemma 1. We first show that for any $\sigma > 2$ there is a sequence of functions $\varphi_n \in S^0(\mathbb{R})$, $n = 0, 1, 2, \ldots$, such that

$$|\varphi_n(z)| \le A_N(1+|x|)^{-N} e^{\sigma|y|}, \tag{a1}$$

$$\ln|\varphi_n(iy)| \ge |y|, \tag{a2}$$

$$\ln|\varphi_n(z)| \le \sigma|y| - n\ln^+(|x|/n) + A, \tag{a3}$$

where $\ln^+ r = \max(0, \ln r)$ and the constants A_N and A are independent of n. (We take $n\ln^+(|x|/n) = 0$ when $n = 0$.)

Such a sequence can be constructed by an iterative procedure used in the theory of quasi-analytic classes and described, for example, in [6], § 1.3. Let $a_0 \ge a_1 \ge \ldots$ be a sequence of positive numbers. Let $H_a(t) = a^{-1}$ for $-a/2 < t < a/2$ and $H_a(t) = 0$ outside this range. We set

$$\omega_n = H_{a_0} * \cdots * H_{a_n}. \tag{a4}$$

Clearly, ω_n is an even nonnegative function supported in the interval $|t| \le (a_0 + \cdots + a_n)/2$. The integral of this function equals 1 because $\int (u * v)\mathrm{d}t = \int u\,\mathrm{d}t \int v\,\mathrm{d}t$. Using the relation

$$(u * H_a)'(t) = \frac{u(t + a/2) - u(t - a/2)}{a},$$

where u is assumed to be a continuous function, we see that $\omega_n^{(k)}$ can be written as a sum of 2^k terms, each of which is a shift of the function $H_{a_k} * \cdots * H_{a_n}/(a_0 \ldots a_{k-1})$. Taking the inequality $|u * v| \le \sup|u| \int |v|\mathrm{d}t$ into account, we obtain

$$|\omega_n^{(k)}| \le \frac{2^k}{a_0 \ldots a_k}, \quad 0 \le k \le n. \tag{a5}$$

We note that $\omega_n \in C_0^{n-1}$, and while the higher derivative $\omega_n^{(n)}$ is only piecewise continuous, the estimate (a5) holds in this case as well. We set $a_0 = 2$, $a_1 = \cdots = a_n = 2/n$. Then

$$|\omega_n^{(k)}| \le \frac{1}{2}\,n^k, \quad k \le n, \quad \int \omega_n\,\mathrm{d}t = 1, \quad \operatorname{supp}\omega_n \subset [-2, 2].$$

Let us consider the convolution $\psi_n = \omega_n * \omega$, where $\omega \in C_0^\infty$ is a smooth nonnegative even function supported in $[-\delta, \delta]$ and having integral 1. The Laplace transform of ψ_n is estimated as follows:

$$|x^k \tilde{\psi}_n(z)| \le \int_{-2-\delta}^{2+\delta} \left| e^{izt}\psi_n^{(k)}(t)\right| \mathrm{d}t \le \begin{cases} C_k\, e^{(2+\delta)|y|} & \text{for all } k; \\ \frac{1}{2} n^k\, e^{(2+\delta)|y|} & \text{for } k \le n. \end{cases} \tag{a6}$$

On the other hand, $\int\limits_{|t|>1-\delta} \psi_n(t)\,\mathrm{d}t \ge \delta$ because $\psi_n \le 1/2$ and $\int \psi_n(t)\,\mathrm{d}t = 1$. Since the function ψ_n is nonnegative and even, we get

$$|\tilde{\psi}_n(iy)| = \int e^{-yt}\,\psi_n(t)\,\mathrm{d}t \ge \int_{t>1-\delta} e^{|y|t}\psi_n(t)\mathrm{d}t \ge \frac{\delta}{2} e^{(1-\delta)|y|}. \tag{a7}$$

Hence the sequence $\varphi_n(z) = (2/\delta)\tilde{\psi}_n(z/(1-\delta))$ possesses all the required properties (a1)–(a3), if δ is chosen so that $(2+\delta)/(1-\delta) < \sigma$.

This sequence is the main tool for proving Lemma 1. Without loss of generality, we can assume that $b = 1$. Let us introduce the auxiliary function

$$H(\xi) = \sup_y \{\ln|\eta(\xi + iy)| - |y|\}, \tag{a8}$$

By (29), it satisfies the inequality

$$H(\xi) \le \ln C_N - N\ln(1+|\xi|) + d(\xi, V). \tag{a9}$$

We first consider the simplest one-dimensional case, when $V = \mathbb{R}_-$ and $d(\xi, V) = \vartheta(\xi)\,|\xi|$, where $\vartheta(x)$ is the Heaviside step function. Let $\Phi_n(z) = \ln|\varphi_n(ez)|$. The function Φ_n is subharmonic according to [5], § II.9.12. As a candidate for the desired function ϱ, we take the upper envelope of the family $\Phi_n(z-\xi) + H(\xi)$, allowing the index n to depend on the point $\xi \in \mathbb{R}$. The functions in this family are locally uniformly bounded from above and hence their upper envelope is also subharmonic (see [5], § II.9.6). Moreover, it obviously dominates $\ln|\eta(z)|$ because relations (a2) and (a8) imply that

$$\sup_\xi \{\Phi_n(z-\xi) + H(\xi)\} \ge \Phi_n(iy) + H(x) \ge \ln|\eta(z)|. \tag{a10}$$

We claim that the second inequality in (30) is ensured by an appropriate choice of $n(\xi)$. If $\xi < 0$, then $d(\xi, \mathbb{R}_-) = 0$ and we can simply set $n(\xi) = 0$, because the property (a1) and the elementary inequality

$$-\ln(1+|x-\xi|) - \ln(1+|\xi|) \le -\ln(1+|x|), \tag{a11}$$

yield that

$$\sup_{\xi<0} \{\Phi_0(z-\xi) + H(\xi)\} \le A'_N + e\sigma|y| - N\ln(1+|x|).$$

In view of (a11), we also have the estimate

$$\varkappa\,\Phi_0(z-\xi) - N\ln(1+|\xi|) \le A''_N + \varkappa e\sigma|y| - N\ln(1+|x|), \tag{a12}$$

which holds for any $\varkappa > 0$ and all ξ and shows that difficulties emerge only from the linear growth of the term $d(\xi, V)$ in (a9).

Suppose that $\xi \ge 0$ and hence $d(\xi, \mathbb{R}_-) = \xi$. Suppose also that $e|x-\xi| > n$. Then

$$n\ln\frac{e|x-\xi|}{n} + ed(x, \mathbb{R}_-) \ge n\ln\frac{e\xi}{n}. \tag{a13}$$

This is obvious for $|x-\xi| > \xi$. When $|x-\xi| \le \xi$, it suffices to use the inequality $\vartheta(x)\,x \ge \xi - |x-\xi|$ and note that the function $n\ln(\lambda/n) - \lambda$ is monotone decreasing in $\lambda \in [n, e\xi]$. Combining (a3) and (a13), we get

$$\Phi_n(z-\xi) + \xi \le A + \sigma e|y| + ed(x, \mathbb{R}_-) - n\ln\frac{e\xi}{n} + \xi.$$

We take $n(\xi)$ to be the integer part of ξ. Then $n\ln(e\xi/n) \ge n > \xi - 1$ and

$$\Phi_{n(\xi)}(z-\xi) + \xi \le A' + \sigma e|y| + ed(x, \mathbb{R}_-). \tag{a14}$$

An analogous inequality holds for $e|x-\xi| \le n$, when $\ln^+(e|x-\xi|/n)$ vanishes. Indeed, in that case $\xi \le \vartheta(x)\,|x| + |x-\xi| \le \vartheta(x)\,|x| + \xi/e$, and hence $\xi \le ed(x, \mathbb{R}_-)$. Thus the inequality (a14) (with an appropriate constant on the right-hand side) holds for all $\xi \ge 0$. Combining this with estimate (a12) and setting $\varkappa = B/(e\sigma) - 1$ in this estimate, we conclude that the upper envelope

$$\varrho(z) = \overline{\lim_{z'\to z}}\, \sup_\xi \{\varkappa\Phi_0(z'-\xi) + \Phi_{n(\xi)}(z'-\xi) + H(\xi)\} \tag{a15}$$

satisfies all the requirements [5].

In the general case of several variables and an arbitrary open cone $V \subset \mathbb{R}^d$, we set $\Phi_n(z) = \sum_{j=1}^d \ln|\varphi_n(e\sqrt{d}\, z_j)|$. Clearly, the inequality (a10) holds. The estimate (a12) is replaced by

$$\varkappa\,\Phi_0(z-\xi) - N\ln(1+|\xi|) \le A'''_N + \varkappa e d\sigma|y| - N\ln(1+|x|), \tag{a16}$$

because $\sum_{j=1}^d |y_j| \le \sqrt{d}\,|y|$. For $x \notin V$ and $e|x-\xi| > n$, we have

$$\sum_{j=1}^d n\ln\left(\frac{e\sqrt{d}}{n}|x_j-\xi_j|\right) + ed(x,V) \ge n\ln\left(\frac{e}{n}d(\xi,V)\right).$$

To prove this, it suffices to use the formulae

$$\sum_{j=1}^d \ln^+|x_j| \ge \ln^+\frac{|x|}{\sqrt{d}}, \qquad d(\xi,V) = \inf_{\xi'\in V}|\xi'-\xi| \le d(x,V) + |x-\xi|.$$

This time we take $n(\xi)$ to be the integral part of $d(\xi,V)$. Then (a14) is replaced by the inequality

$$\Phi_{n(\xi)}(z-\xi) + d(\xi,V) \le A'' + \sigma ed|y| + ed(x,V),$$

which holds for all x. Combining this inequality with (a16), we conclude that the conditions (30) are fulfilled for the plurisubharmonic function defined by (a15) with $\xi \in \mathbb{R}^d$ and $\varkappa = B/(e\sigma d) - 1$. Lemma 1 is proved. □

Appendix 2.

Suppose that $1 < \alpha' < \alpha$. We now return to (a4) and set $a_0 = a_1 = 1$ and $a_k = (k-1)^{k-1)\alpha'}/k^{k\alpha'}$ for $k > 1$. The series $\sum a_k$ converges because $(k-1)^{k-1}/k^k \le 1/k$. By Theorem 1.3.5 of [6], the corresponding sequence (a4) tends to a smooth nonnegative even function as $n \to \infty$. This function is compactly supported, and its k th derivative is bounded by $2^k k^{k\alpha'}$. By a scaling transformation it can be converted into a function γ such that

$$|\gamma^{(k)}| \le C_\epsilon \epsilon^k k^{k\alpha}, \qquad \operatorname{supp}\gamma \subset [-1/2, 1/2],$$

where ϵ is arbitrarily small. Let $g_\alpha(x) = \tilde{\gamma}^2(x)$, where $\tilde{\gamma}$ is the Laplace transform of γ. Then

$$|g_\alpha(x+iy)| \le e^{-2|x|^{1/\alpha}+|y|}, \quad g_\alpha(x) \ge 0, \quad g_\alpha(0) > 0. \tag{a17}$$

Proof of Theorem 9. We first consider the special case of the closed half-plane $\mathbb{R}\times\bar{\mathbb{R}}_-$ in $\mathbb{R}^2$ (the general case can easily be reduced to this one). Using g_α, we can construct functions $f_1 \in S^0(\mathbb{R})$ and $f_2 \in S^0(\mathbb{R}_-)$ such that $f_1 \otimes f_2 \in S^0(\mathbb{R}, \bar{\mathbb{R}}_-) = S^0(\mathbb{R}\times\mathbb{R}_-)$ and $f_1\otimes f_2 \notin S^\beta(\mathbb{R}\times\bar{\mathbb{R}}_-)$ for any $\beta \in [0,1)$. We set

$$f_1(x) = \int e^{-|\xi|^{1/\alpha}} g_\alpha(x-\xi)\,\mathrm{d}\xi.$$

This convolution can be analytically continued to whole of $\mathbb{C}$ and belongs to $S^0_\alpha(\mathbb{R})$. Indeed, using the triangle inequality for the metric $|x-\xi|^{1/\alpha}$, we obtain

$$|f_1(x+iy)| \le \int e^{-|x-\xi|^{1/\alpha}-2|\xi|^{1/\alpha}+|y|}\,\mathrm{d}\xi \le C\,e^{-|x|^{1/\alpha}+|y|},$$

where $C = \int e^{-|\xi|^{1/\alpha}}\,\mathrm{d}\xi$. In addition, we have the lower estimate

$$f_1(x) \ge \int_{-1}^{+1} e^{-|x-\xi|^{1/\alpha}} g_\alpha(\xi)\,\mathrm{d}\xi \ge e^{-(|x|+1)^{1/\alpha}}\int_{-1}^{+1} g_\alpha(\xi)\,\mathrm{d}\xi \ge c\,e^{-|x|^{1/\alpha}}. \tag{a18}$$

[5] Taking the upper limit ensures the upper semicontinuity of the resulting function and enters into the definition [5] of upper envelope.

Furthermore, let $1 < \alpha' < \alpha$ and let

$$f_2(x) = \int_0^\infty e^{\xi^{1/\alpha'}} g_{\alpha'}(x-\xi)\,\mathrm{d}\xi.$$

In an analogous way, it is easy to verify that

$$|f_2(x+iy)| \le C'\, e^{|x|^{1/\alpha'}+|y|}. \tag{a19}$$

For $x > 1$, we have the estimate

$$f_2(x) = \int_{-\infty}^x e^{(x-\xi)^{1/\alpha'}} g_{\alpha'}(\xi)\,\mathrm{d}\xi \ge e^{(x-1)^{1/\alpha'}} \int_{-1}^{+1} g_{\alpha'}(\xi)\,\mathrm{d}\xi \ge c'\, e^{x^{1/\alpha'}}. \tag{a20}$$

If $x < 0$ and $\xi > 0$, then $|x-\xi| = |x| + |\xi|$. Using the inequality $2(|x|+|\xi|)^{1/\alpha'} \ge |2x|^{1/\alpha'} + |2\xi|^{1/\alpha'}$, we obtain

$$|f_2(x+iy)| \le C''\, e^{-|2x|^{1/\alpha'}+|y|}, \qquad x \in \mathbb{R}_-. \tag{a21}$$

The estimates (19) and (21) imply that $f_2 \in S^0(\mathbb{R}_-)$ and, therefore,

$$(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1) f_2(x_2) \in S^0(\mathbb{R} \times \mathbb{R}_-).$$

On the other hand, the lower estimates (a18) and (a20) show that the function $f_1 \otimes f_2$ increases to infinity along any real ray in the half-plane $x_2 > 0$. Hence it does not belong to any of the $S^\beta(U)$, where $U \Supset \mathbb{R} \times \bar{\mathbb{R}}_-$. Setting

$$(v, f) = \int_1^\infty f(x_1, x_1^{\alpha'/\alpha})\,\mathrm{d}x_1,$$

we obtain a simple example of a functional in ${S'}^0(\mathbb{R} \times \bar{\mathbb{R}}_-)$ which does not belong to ${S'}^0(\mathbb{R}, \bar{\mathbb{R}}_-)$. The function $f_1 \otimes f_2$ constructed above is bounded below by a positive constant on the path of integration $x_2 = x_1^{\alpha'/\alpha}$, $x_1 > 1$. Therefore v has no continuous extension to $S^0(\mathbb{R} \times \mathbb{R}_-)$, nor to any $S^\beta(\mathbb{R} \times \mathbb{R}_-)$, $\beta > 0$. Indeed, $f_1 \otimes f_2$ can be approximated (in the topology of $S^0(\mathbb{R} \times \mathbb{R}_-)$) by positive functions $f_\nu = f_1 \otimes (g_\nu\, f_2) \in S^0_\alpha(\mathbb{R}^2)$, where $g_\nu(x_2) = g_{\alpha''}(x_2/\nu)$, $\alpha'' < \alpha'$, and the normalization condition $g_{\alpha''}(0) = 1$ is assumed. Clearly we have $(v, f_\nu) \to \infty$ as $\nu \to \infty$, so there is no continuous extension.

Now, let K_1 and K_2 be closed cones in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, where $d_1 \ge 1$ and $d_2 \ge 1$. Suppose that $K_1 \ne \{0\}$ and $K_2 \ne \mathbb{R}^{d_2}$. We claim that $S^\beta(K_1 \times K_2)$ does not contain $S^0(K_1, K_2)$ and does not even contain the smaller space $S^0(\mathbb{R}^{d_1}, K_2)$. Indeed, assume that the first basis vector e_1^1 in $\mathbb{R}^{d_1}$ belongs to K_1 and the basis vector e_2^1 in $\mathbb{R}^{d_2}$ does not belong to K_2. Let h_1 be a function in $S^0_\alpha(\mathbb{R}^{d_1-1})$ such that $h_1(0) \ne 0$ and replace f_1 by $f_1 \otimes h_1$ in the above construction. Clearly, $f_1 \otimes h_1 \in S^0_\alpha(\mathbb{R}^{d_1})$. We also replace f_2 by $f_2 \otimes h_2$, where $h_2 \in S^0_{\alpha''}(\mathbb{R}^{d_2-1})$ and $h_2(0) \ne 0$. It is easy to see that $f_2 \otimes h_2 \in S^0_{\alpha'}(U)$, where U is an open cone in $\mathbb{R}^{d_2}$ defined by the inequality $(1+\theta)x_2^1 < |x_2|$. Clearly, $K_2 \setminus \{0\}$ is contained in this cone if $\theta > 0$ is small enough. Therefore, the function $f_1 \otimes h_1 \otimes f_2 \otimes h_2$ belongs to $S^0_\alpha(\mathbb{R}^{d_1} \times U)$, while none of the spaces $S^\beta(K_1 \times K_2)$, $0 \le \beta < 1$, contains this function, as is evident from (a18) and (a20). In complete analogy with what was done above, we define a functional v by integrating test functions along the curve $x_2^1 = (x_1^1)^{\alpha'/\alpha}$, $x_1 > 1$, in the plane $\{e_1^1, e_2^1\}$. This functional is carried by the ray $\{\lambda e_1^1 \mid \lambda \ge 0\}$ lying on the boundary of $K_1 \times K_2$, but v does not belong to ${S'}^\beta(K_1, K_2)$, nor even to ${S'}^\beta(\mathbb{R}^{d_1}, K_2)$. This completes the proof. □

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